Neyman Pearson Lemma

Jacques Amsel

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1 Statement

If a simple null and alternative hypothesis is used ($H_o : \theta = \theta_o$ vs $H_a : \theta = \theta_a$) with a specific α, any other test with $\alpha^* \leq \alpha$ will have $1 - \beta^* \leq LR$

$$LR = \frac{L(\theta_o)}{L(\theta_a)} < k$$

(Note: here, k is related to the rejection region of the test)

2 Proof

A test for $H_o : \theta = \theta_o$ vs $\theta = \theta_a$ means to use

$$d(y) = \begin{cases} 0 & H_o \text{ fails to be rejected} \\ 1 & H_o \text{ is rejected} \end{cases}$$

$$\begin{aligned}
&E[d(Y)|H_o \text{ true}] \\
&\quad = 1 \cdot P(H_o \text{ rejected}|H_o \text{ true}) + 0 \cdot P(H_0 \text{ fails to be rejected}|H_o \text{ true}) \\
&\quad = \alpha \\
&E[d(Y)|H_a \text{ true}] \\
&\quad = 1 \cdot P(H_o \text{ rejected}|H_a \text{ true}) + 0 \cdot P(H_o \text{ fails to be rejected}|H_o \text{ true}) \\
&\quad = P(H_o \text{ rejected}|H_a \text{ true}) = 1 - P(H_o \text{ fails to be rejected}|H_a \text{ true}) \\
&\quad = 1 - \beta
\end{aligned}$$

let $d(Y)$ be associated with the LR test (i.e. $H_o : \theta = \theta_o, etc$),

$$d(Y) = 1 \Rightarrow \frac{L(\theta_o)}{L(\theta_a)} < k \Rightarrow L(\theta_o) < k \cdot L(\theta_a)$$

let $d^*(Y)$ be the decision function of any other test such that

$$\alpha^* = E[d^*(Y)|H_o \text{ true}] \leq E[d(Y)|H_o \text{ true}] = \alpha \tag{1}$$

We must show that

$$1 - \beta^* = E[d^*(Y)|H_a \text{ true}] \leq E[d(Y)|H_a \text{ true}] = 1 - \beta$$

If $d(Y) = 1 \Rightarrow k \cdot L(\theta_a) > L(\theta_o) \Rightarrow 0 < k \cdot L(\theta_a) - L(\theta_o)$

If $d(Y) = 0 \Rightarrow k \cdot L(\theta_a) < L(\theta_o) \Rightarrow 0 > k \cdot L(\theta_a) - L(\theta_o)$

$$d^*[k \cdot L(\theta_a) - L(\theta_o)] \leq d(Y)[k \cdot L(\theta_a) - L(\theta_o)] \tag{2}$$

START NOT IN PROOF:

Why?

If $d(Y) = 1:$

$\frac{L(\theta_o)}{L(\theta_a)} < k \Rightarrow k \cdot L(\theta_a) - L(\theta_o) > 0$ (proven previously)

Since $d^*(Y) = 0$ or 1 and $d(Y) = 1$

$d^*(Y) \cdot positive \leq d(Y) \cdot positive$

If $d(Y) = 0:$

$\frac{L(\theta_o)}{L(\theta_a)} > k \Rightarrow k \cdot L(\theta_a) - L(\theta_o) < 0$ (proven previously)

Since $d^*(Y) = 0$ or 1 and $d(Y) = 1$

$d^*(Y) \cdot negative \leq d(Y) \cdot negative$

END NOT IN PROOF

Integrate (2) on both sides dy.

$$\int (k \cdot d^*(y)L(\theta_a) - d^*(y)L(\theta_o))dy \leq \int (k \cdot d(Y)L(\theta_a) - d(y)L(\theta_o))dy$$

Since $L(\theta_x) = f(y|\theta_x)$

$$\int (k \cdot d^*(y)f(y|\theta_a) - d^*(y)f(y|\theta_o))dy \leq \int (k \cdot d(Y)f(y|\theta_a) - d(y)f(y|\theta_o))dy$$

$$k \int d^*(y)f(y|\theta_a)dy - \int d^*(y)f(y|\theta_o)dy \leq k \int d(Y)f(y|\theta_a)dy - \int d(y)f(y|\theta_o)dy$$

$$kE[d^*(y)|H_a \text{ true}] - E[d^*(y)|H_o \text{ true}] \le kE[d(Y)|H_a \text{ true}] - E[d(y)|H_o \text{ true}]$$

$$k(1-\beta^*) - \alpha^* \le k(1-\beta) - \alpha$$

$$k(1-\beta^*) - \alpha^* \le k(1-\beta) - \alpha$$

$$\alpha - \alpha^* \le k(\beta^* - \beta)$$

Since, from (1), $\alpha \ge \alpha^*$

$$0 \le k(\beta^* - \beta) \Rightarrow \beta \le \beta^* \Rightarrow 1 - \beta \ge 1 - \beta^*$$